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Permutations and the loop

T.W. Brown

Centre for Research in String Theory, Department of Physics,
Queen Mary, University of London,
Mile End Road, London E1 4NS, U.K.
E-mail: t.w.brown@qmul.ac.uk

Abstract: We consider the one-loop two-point function for multi-trace operators in the $\mathrm{U}(2)$ sector of $\mathcal{N}=4$ supersymmetric Yang-Mills at finite N. We derive an expression for it in terms of $\mathrm{U}(N)$ and S_{n+1} group theory data, where n is the length of the operators. The Clebsch-Gordan operators constructed in [1], which are diagonal at tree level, only mix at one loop if you can reach the same $(n+1)$-box Young diagram by adding a single box to each of the n-box Young diagrams of their $\mathrm{U}(N)$ representations (which organise their multi-trace structure). Similar results are expected for higher loops and for other sectors of the global symmetry group.

Keywords: AdS-CFT Correspondence, 1/N Expansion, Supersymmetric gauge theory.

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1. Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills has three complex scalars transforming in the adjoint representation of the gauge group $\mathrm{U}(N)$. We focus on operators built out of two of the complex scalars, X and Y, which is a $\mathrm{U}(2) \subset \mathrm{SU}(4) \subset \mathrm{PSU}(2,2|4)$ subsector of the full global symmetry group of the theory. Their basic correlators are given in terms of their $\mathrm{U}(N)$ fundamental and antifundamental indices

$$\begin{aligned} \left\langle X^{\dagger}{}^i_j(x)\, X^k_l(0) \right\rangle &= \left\langle Y^{\dagger}{}^i_j(x)\, Y^k_l(0) \right\rangle = \frac{1}{x^2}\, \delta^i_l\, \delta^k_j \\ \left\langle X^{\dagger}{}^i_j(x)\, Y^k_l(0) \right\rangle &= 0 \end{aligned} \tag{1.1}$$

From here onwards we will drop the spacetime dependence of the correlators and focus on the combinatorial parts. We will use a convention whereby $\langle \cdots \rangle$ means the tree-level correlator where we Wick contract with (1.1).

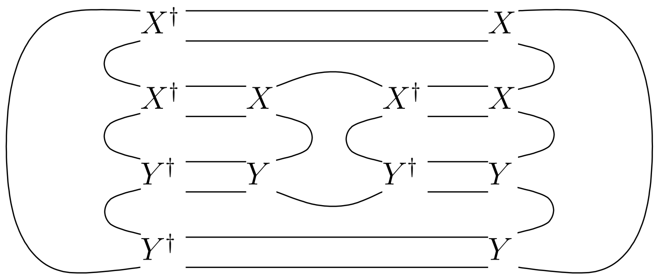

Figure 1: A planar one-loop diagram for a part of the two-point function between $\mathrm{tr}(XXYY)$ and $\mathrm{tr}(X^\dagger X^\dagger Y^\dagger Y^\dagger)$ with the $\mathrm{tr}(YXX^\dagger Y^\dagger)$ effective vertex; note this leading N^{4+1} behaviour

We can build gauge-invariant operators by taking traces such as $\mathrm{tr}(Y)\,\mathrm{tr}(XYX)$ or $\mathrm{tr}(XXYY)$. These can be written by letting permutations act on the gauge indices

$$\mathrm{tr}(Y)\,\mathrm{tr}(XYX) = X^{i_1}_{i_4} X^{i_2}_{i_1} Y^{i_3}_{i_3} Y^{i_4}_{i_2} = X^{i_1}_{i_{\alpha(1)}} X^{i_2}_{i_{\alpha(2)}} Y^{i_3}_{i_{\alpha(3)}} Y^{i_4}_{i_{\alpha(4)}} \equiv \mathrm{tr}(\alpha\, XXYY) \tag{1.2}$$

Here $\alpha = (142)$ is an element of the symmetric group S_4 of permutations of four objects.

In this paper we derive an expression for the one-loop two-point function of these operators in terms of this group-theoretic language. In essence all this requires is that we follow permutations and double-line index loops [2] carefully. We make extensive use of the representation theory methods developed for the $\mathrm{U}(1)$ sector in [3] and the diagrammatic techniques introduced in [4].

At tree level the correlator in terms of permutations is [1]

$$\left\langle \mathrm{tr}(\alpha_2\, X^{\dagger\mu} Y^{\dagger\nu})\ \mathrm{tr}(\alpha_1\, X^{\mu} Y^{\nu}) \right\rangle = \frac{1}{\mu!\nu!} \sum_{\sigma,\tau \in S_\mu \times S_\nu} \sum_{T \vdash n} \chi_T(\sigma^{-1}\alpha_1\sigma\ \tau^{-1}\alpha_2\tau)\mathrm{Dim}T \tag{1.3}$$

Here X^μ just means μ copies of X (μ is a power not an index) and similarly for Y. $S_\mu \times S_\nu$ is the subgroup of the symmetric group $S_{\mu+\nu}$ that doesn't mix the first μ items with the last ν, reflecting the fact that X does not mix with Y when we Wick contract with (1.1).[1] We sum over all $n \equiv \mu + \nu$ box Young diagrams T with at most N rows, each of which labels an irreducible representation both of S_n and of $\mathrm{U}(N)$. This Schur-Weyl duality of the irreducible representations of S_n and $\mathrm{U}(N)$ follows because they have a commuting action on $V_N^{\otimes n}$ where V_N is the fundamental representation space of $\mathrm{U}(N)$. χ_T is an S_n character and $\mathrm{Dim}T$ is the dimension of the $\mathrm{U}(N)$ representation. Because T has n boxes its leading large N behaviour is $\mathrm{Dim}T \sim kN^n$ (see identity (A.1)).

In [1] a basis $\mathcal{O}[\Lambda,\mu,\nu,\beta;R;\tau]$ was found that diagonalises this tree-level two-point function. $[\Lambda,\mu,\nu,\beta]$ labels the $\mathrm{U}(2)$ representation and state while R labels the $\mathrm{U}(N)$ representation which organises the multi-trace structure.[2]

[1]This expression for the tree level correlator is a little redundant because we can absorb the τ sum into the σ sum; we have written it like this to emphasis the comparison with the one-loop case.

[2]The operator as a whole is a $\mathrm{U}(N)$ singlet since it is gauge-invariant.

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At one loop we get corrections from the self-energy, the scalar four-point vertex and the exchange of a gluon. Cancellations among these corrections mean that the one-loop correlator is given by an effective vertex [5, 6]

$$\Big\langle \mathrm{tr}(\alpha_2\, X^{\dagger\mu} Y^{\dagger\nu}) : \mathrm{tr}([X,Y][X^\dagger,Y^\dagger]) : \mathrm{tr}(\alpha_1\, X^\mu Y^\nu) \Big\rangle \tag{1.4}$$

For convenience we have dropped a $-\frac{g_{\mathrm{YM}}^2}{8\pi}$ prefactor and the spacetime dependence $\log(x\Lambda)^{-2}/x^{2n}$ for some cutoff Λ. The expression betwen colons :: is normal-ordered so that no contractions within the colons is allowed. In sections 2 and 3 we derive an expression for this one-loop correlator in terms of permutations

$$\frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma,\tau\in S_\mu\times S_\nu}\ \sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2)\sum_{T\vdash n+1}\chi_T(\rho_1\,\sigma^{-1}\alpha_1\sigma\,\rho_2\,\tau^{-1}\alpha_2\tau)\mathrm{Dim}T \tag{1.5}$$

Compare this with (1.3). Now T has $n+1$ boxes and χ_T is a character of S_{n+1}. For large N the leading behaviour is $\mathrm{Dim}T \sim kN^{n+1}$, which is what we expect for the one-loop result (see for example figure 1). $h(\rho_1,\rho_2)$ only takes non-zero values on a few permutations of the μ, n and $n+1$ indices (it is given in full in equation (2.12)); it encodes the commutators in (1.4).

We also derive a similar expression for the one-loop dilatation operator.

We find that the Clebsch-Gordan basis $\mathcal{O}[\Lambda,\mu,\nu,\beta;R;\tau]$ has constrained mixing at one loop. If two operators are in the same U(2) representation and state, then if their U(N) representations R_1 and R_2 are different they only mix if we can add a box to each Young diagram to get the same U(N) representation with $n+1$ boxes T. For example $R_1 = \yng(2,2)$ and $R_2 = \yng(3,1)$ mix because we can get them both by knocking a single box off $T = \yng(3,2)$. In other words, when we restrict the representation T of S_{n+1} to its S_n subgroup, R_1 and R_2 must both appear in the reduction. This mixing is analysed in section 4. A detailed look at the U(2) representation $\Lambda = \yng(2,2)$ operators is given in appendix E.

Extensions to higher loops and the rest of the global symmetry are discussed in section 5.

Appendix A covers some group theory conventions and formulae; appendix B briskly introduces the diagrammatic formalism we use; appendix C revises the construction of the representing matrices for the symmetric group.

2. Dilatation operator

Given that $\Big\langle X^{\dagger}{}^i_j\, X^k_l \Big\rangle = \tilde{X}^i_j\, X^k_l = \delta^i_l \delta^k_j$ where $\tilde{X}^i_j = \frac{d}{dX^j_i}$ we can get the one-loop correlator by first acting on $\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)$ with the one-loop dilatation operator [6–9]

$$\Delta^{(1)} = \mathrm{tr}([X,Y][\tilde{X},\tilde{Y}]) \tag{2.1}$$

As a warm-up consider the action of $\tilde{X}^a_b$ on

$$X^{i_1}_{j_1}\cdots X^{i_n}_{j_n} \tag{2.2}$$

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By the product rule we get

$$\left(\delta^a_{j_1}\delta^{i_1}_b\right)\ X^{i_2}_{j_2}\cdots X^{i_n}_{j_n}\ +\ X^{i_1}_{j_1}\ \left(\delta^a_{j_2}\delta^{i_2}_b\right)\ X^{i_3}_{j_3}\cdots X^{i_n}_{j_n}\ +\ \cdots \tag{2.3}$$

To write this down in terms of permutations we shuffle around the δ's with $\sigma \in S_n$ so that the derivative only ever acts on the final index

$$\frac{1}{(n-1)!}\sum_{\sigma\in S_n}\left(\delta^a_{j_{\sigma(n)}}\delta^{i_{\sigma(n)}}_b\right)\ X^{i_{\sigma(1)}}_{j_{\sigma(1)}}\cdots X^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}} \tag{2.4}$$

We divide by $(n-1)!$ because summing over all of S_n is redundant.[3]

It is a small step now to the action of $\tilde{X}^a_b\tilde{Y}^c_d$ on

$$X^{i_1}_{j_1}\cdots X^{i_\mu}_{j_\mu}Y^{i_{\mu+1}}_{j_{\mu+1}}\cdots Y^{i_{\mu+\nu}}_{j_{\mu+\nu}} \tag{2.5}$$

We get

$$\frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\left(\delta^a_{j_{\sigma(\mu)}}\delta^{i_{\sigma(\mu)}}_b\right)\left(\delta^c_{j_{\sigma(\mu+\nu)}}\delta^{i_{\sigma(\mu+\nu)}}_d\right)X^{i_{\sigma(1)}}_{j_{\sigma(1)}}\cdots X^{i_{\sigma(\mu-1)}}_{j_{\sigma(\mu-1)}}\ Y^{i_{\sigma(\mu+1)}}_{j_{\sigma(\mu+1)}}\cdots Y^{i_{\sigma(\mu+\nu-1)}}_{j_{\sigma(\mu+\nu-1)}}$$

Next we relabel indices $i_{\sigma(k)} \to p_k$ and $j_{\sigma(k)} \to q_k$ for $k \in \{1,\dots\mu-1,\mu+1,\dots\mu+\nu-1\}$. This amounts to writing $X^{i_{\sigma(k)}}_{j_{\sigma(k)}} = \delta^{i_{\sigma(k)}}_{p_k}\delta^{q_k}_{j_{\sigma(k)}}X^{p_k}_{q_k}$, which is just a book-keeping exercise.[4]

$$\begin{aligned}&\frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\left(\delta^a_{j_{\sigma(\mu)}}\delta^{i_{\sigma(\mu)}}_b\right)\left(\delta^c_{j_{\sigma(\mu+\nu)}}\delta^{i_{\sigma(\mu+\nu)}}_d\right)\\ &\qquad\delta^{i_{\sigma(1)}}_{p_1}\cdots\delta^{i_{\sigma(\mu-1)}}_{p_{\mu-1}}\ \delta^{i_{\sigma(\mu+1)}}_{p_{\mu+1}}\cdots\delta^{i_{\sigma(\mu+\nu-1)}}_{p_{\mu+\nu-1}}\ \ \delta^{q_1}_{j_{\sigma(1)}}\cdots\delta^{q_{\mu-1}}_{j_{\sigma(\mu-1)}}\ \delta^{q_{\mu+1}}_{j_{\sigma(\mu+1)}}\cdots\delta^{q_{\mu+\nu-1}}_{j_{\sigma(\mu+\nu-1)}}\\ &\qquad\quad X^{p_1}_{q_1}\cdots X^{p_{\mu-1}}_{q_{\mu-1}}\ Y^{p_{\mu+1}}_{q_{\mu+1}}\cdots Y^{p_{\mu+\nu-1}}_{q_{\mu+\nu-1}}\end{aligned} \tag{2.6}$$

Now let's contract some indices. We're not interested in the gauge-covariant operator (2.5); we'd like to know about $\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)$, which means setting $j_m = i_{\alpha_1(m)}$. Also we need to contract the indices of the dilatation operator $\mathrm{tr}([X,Y][\tilde{X},\tilde{Y}])$

$$\begin{aligned}&\mathrm{tr}(XY\tilde{X}\tilde{Y})-\mathrm{tr}(YX\tilde{X}\tilde{Y})-\mathrm{tr}(XY\tilde{Y}\tilde{X})+\mathrm{tr}(YX\tilde{Y}\tilde{X})\\ &\quad=X^{p_\mu}_{q_\mu}Y^{p_{\mu+\nu}}_{q_{\mu+\nu}}\tilde{X}^a_b\tilde{Y}^c_d\Big(\delta^{q_\mu}_{p_{\mu+\nu}}\delta^{q_{\mu+\nu}}_a\delta^b_c\delta^d_{p_\mu}-\delta^{q_\mu}_a\delta^{q_{\mu+\nu}}_{p_\mu}\delta^b_c\delta^d_{p_{\mu+\nu}}-\delta^{q_\mu}_{p_{\mu+\nu}}\delta^{q_{\mu+\nu}}_c\delta^b_{p_\mu}\delta^d_a+\delta^{q_\mu}_c\delta^{q_{\mu+\nu}}_{p_\mu}\delta^b_{p_{\mu+\nu}}\delta^d_a\Big)\end{aligned} \tag{2.7}$$

This all looks frightful, but let's take the first term of the one-loop dilatation operator and work it out

$$\begin{aligned}\mathrm{tr}(XY\tilde{X}\tilde{Y})\,[\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)] &= \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\delta^{q_{\mu+\nu}}_{i_{\alpha_1\sigma(\mu)}}\ \delta^{i_{\sigma(\mu)}}_{i_{\alpha_1\sigma(\mu+\nu)}}\delta^{i_{\sigma(\mu+\nu)}}_{p_\mu}\delta^{q_\mu}_{p_{\mu+\nu}}\\ \delta^{i_{\sigma(1)}}_{p_1}\cdots\delta^{i_{\sigma(\mu-1)}}_{p_{\mu-1}}\ \delta^{i_{\sigma(\mu+1)}}_{p_{\mu+1}}\cdots\delta^{i_{\sigma(\mu+\nu-1)}}_{p_{\mu+\nu-1}}&\quad\ \delta^{q_1}_{i_{\alpha_1\sigma(1)}}\cdots\delta^{q_{\mu-1}}_{i_{\alpha_1\sigma(\mu-1)}}\ \delta^{q_{\mu+1}}_{i_{\alpha_1\sigma(\mu+1)}}\cdots\delta^{q_{\mu+\nu-1}}_{i_{\alpha_1\sigma(\mu+\nu-1)}}\\ &\quad\ X^{p_1}_{q_1}\cdots X^{p_\mu}_{q_\mu}\ Y^{p_{\mu+1}}_{q_{\mu+1}}\cdots Y^{p_{\mu+\nu}}_{q_{\mu+\nu}}\end{aligned} \tag{2.8}$$

[3]It would be more economical to sum over $\sigma \in \mathrm{Sym}(n)$, the symmetry group of an n-cycle, in which case we would not have to divide by $(n-1)!$, but this is not necessary for our purposes.

[4]We advise the reader to glance over appendix B for the delta function and diagrammatic techniques used here.

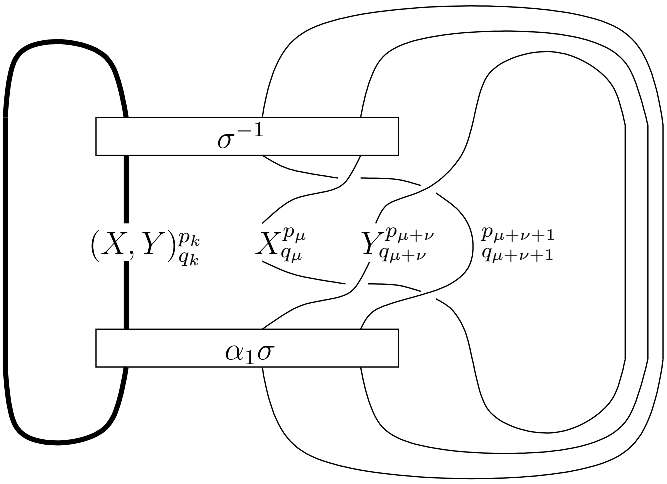


Figure 2: The first term $\mathrm{tr}(XY\tilde{X}\tilde{Y})$ of the one-loop dilatation operator acting on $\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)$; k labels the indices in $\{1,\dots\mu-1,\mu+1,\dots\mu+\nu-1\}$ and these delta function strands are grouped together into a single thick strand; the μ, $\mu+\nu$ and $\mu+\nu+1$ strands are drawn separately

Although this still looks rather ghastly, we can see some similarities emerging between the terms from the dilatation operator on the first line and those on the second line from the Wick contractions. They become clear if we introduce an extra index $\mu+\nu+1$ and split out the deltas $\delta^{q_\mu}_{p_{\mu+\nu}} = \delta^{q_\mu}_{i_{\mu+\nu+1}}\delta^{i_{\mu+\nu+1}}_{p_{\mu+\nu}}$ and $\delta^{i_{\sigma(\mu)}}_{i_{\alpha_1\sigma(\mu+\nu)}} = \delta^{i_{\sigma(\mu)}}_{p_{\mu+\nu+1}}\delta^{p_{\mu+\nu+1}}_{q_{\mu+\nu+1}}\delta^{q_{\mu+\nu+1}}_{i_{\alpha_1\sigma(\mu+\nu)}}$. The expression is now more pleasing

$$\begin{aligned}\mathrm{tr}(XY\tilde{X}\tilde{Y})\,[\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)] = \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu} & X^{p_1}_{q_1}\cdots X^{p_\mu}_{q_\mu}\;\; Y^{p_{\mu+1}}_{q_{\mu+1}}\cdots Y^{p_{\mu+\nu}}_{q_{\mu+\nu}}\;\delta^{p_{\mu+\nu+1}}_{q_{\mu+\nu+1}} \\ & \delta^{i_{\sigma(1)}}_{p_1}\cdots\delta^{i_{\sigma(\mu-1)}}_{p_{\mu-1}}\delta^{i_{\sigma(\mu+\nu)}}_{p_\mu}\delta^{i_{\sigma(\mu+1)}}_{p_{\mu+1}}\cdots\delta^{i_{\sigma(\mu+\nu-1)}}_{p_{\mu+\nu-1}}\delta^{i_{\mu+\nu+1}}_{p_{\mu+\nu}}\delta^{i_{\sigma(\mu)}}_{p_{\mu+\nu+1}} \\ & \delta^{q_1}_{i_{\alpha_1\sigma(1)}}\cdots\delta^{q_{\mu-1}}_{i_{\alpha_1\sigma(\mu-1)}}\delta^{q_\mu}_{i_{\mu+\nu+1}}\delta^{q_{\mu+1}}_{i_{\alpha_1\sigma(\mu+1)}}\cdots\delta^{q_{\mu+\nu-1}}_{i_{\alpha_1\sigma(\mu+\nu-1)}}\delta^{q_{\mu+\nu}}_{i_{\alpha_1\sigma(\mu)}}\delta^{q_{\mu+\nu+1}}_{i_{\alpha_1\sigma(\mu+\nu)}} \end{aligned} \quad (2.9)$$

Introducing the extra index allows us to draw this diagrammatically as a trace of a series of operations on the strands, see figure 2. This was not possible with the expression in (2.8). Converting the diagram back to a formula we get

$$\begin{aligned}&\mathrm{tr}(XY\tilde{X}\tilde{Y})\,[\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)] \\ &\quad = \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\mathrm{tr}\left((\mu,\mu+\nu+1,\mu+\nu)\sigma^{-1}\alpha_1\sigma(\mu,\mu+\nu+1,\mu+\nu)X^\mu Y^\nu\mathbb{I}_N\right)\end{aligned} \quad (2.10)$$

$\mathbb{I}_N$ is a single $\mathrm{U}(N)$ identity matrix and $(\mu,\mu+\nu+1,\mu+\nu)$ is a 3-cycle permutation in S_{n+1}.

If we include the other terms in the one-loop dilatation operator (2.7) then we get

$$\begin{aligned}&\mathrm{tr}([X,Y][\tilde{X},\tilde{Y}])\,[\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)] \\ &\quad = \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2)\,\mathrm{tr}(\rho_1\,\sigma^{-1}\alpha_1\sigma\,\rho_2\,X^\mu Y^\nu\mathbb{I}_N)\end{aligned} \quad (2.11)$$

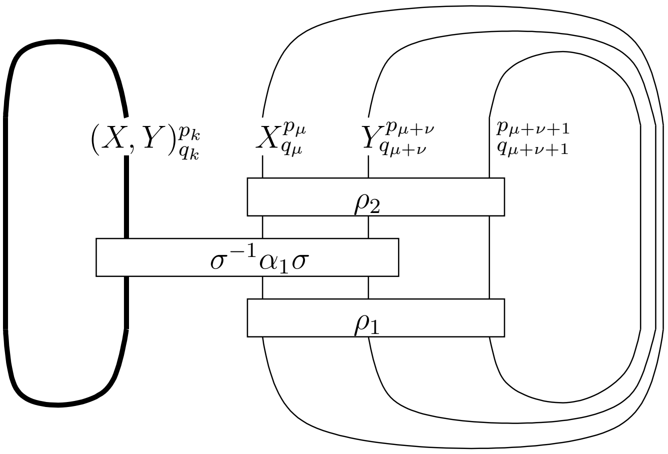


Figure 3: The general diagram for any of the four terms of the one-loop dilatation operator

See figure 3 for the diagram for general ρ_1, ρ_2. h takes non-zero values on

$$\begin{aligned} h((\mu,n+1,n),(\mu,n+1,n)) &= 1 \\ h((\mu,n+1),(n,n+1)) &= -1 \\ h((n,n+1),(\mu,n+1)) &= -1 \\ h((\mu,n,n+1),(\mu,n,n+1)) &= 1 \end{aligned} \tag{2.12}$$

We can write this in a more symmetric fashion that better reflects the commutator structure of the one-loop dilatation operator

$$\begin{aligned} h(\ (\mu,n+1),\ \ (n,n+1)\ ) &= -1 \\ h(\ (\mu,n)\ (\mu,n+1),\ \ (n,n+1)\ (\mu,n)\ ) &= 1 \\ h(\ (\mu,n)\ (\mu,n+1)\ (\mu,n),\ \ (\mu,n)\ (n,n+1)\ (\mu,n)\ ) &= -1 \\ h(\ (\mu,n+1)\ (\mu,n),\ \ (\mu,n)\ (n,n+1)\ ) &= 1 \end{aligned} \tag{2.13}$$

We will use this later.

We can see that this extra index gives an enhancement by a factor of N when a loop forms, see figure 4. This happens when $\sigma^{-1}\alpha_1\sigma$ maps $\mu+\nu \mapsto \mu$ or $\mu \mapsto \mu+\nu$, i.e. when X and Y are next to each other in a trace $\mathrm{tr}(\cdots XY \cdots)$. This is well-studied in the planar context where this contribution dominates and the model is exactly solvable by the Bethe Ansatz (see for example [10–12]). In the non-planar context the trace structure of the operator is still modified when $\sigma^{-1}\alpha_1\sigma$ does not satisfy this condition, and traces can split and join (see for example [13]).

3. One-loop correlator

To get the one-loop correlator we take the tree-level correlator of $\mathrm{tr}(\alpha_2\, X^{\dagger\mu} Y^{\dagger\nu})$ with the

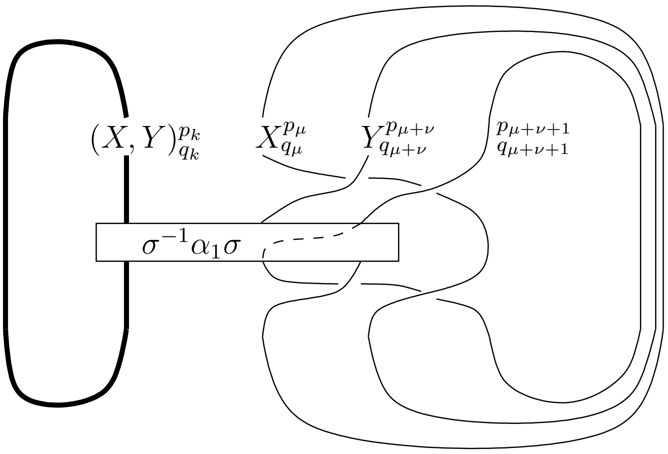


Figure 4: An example of how the extra index allows an index loop to form, giving an N enhancement

image of $\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)$ under the one-loop dilatation operator

$$\begin{aligned}&\Big\langle \mathrm{tr}(\alpha_2\, X^{\dagger\mu}Y^{\dagger\nu}) : \mathrm{tr}([X,Y][X^\dagger,Y^\dagger]) : \mathrm{tr}(\alpha_1\, X^\mu Y^\nu)\Big\rangle \\ &\quad = \Big\langle \mathrm{tr}(\alpha_2\, X^{\dagger\mu}Y^{\dagger\nu})\, \mathrm{tr}([X,Y][\tilde{X},\tilde{Y}])\, [\mathrm{tr}(\alpha_1\, X^\mu Y^\nu)]\Big\rangle \\ &\quad = \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2) \\ &\qquad \Big\langle X^{\dagger j_1}_{\ \ j_{\alpha_2(1)}}\cdots Y^{\dagger j_n}_{\ \ j_{\alpha_2(n)}}\quad X^{i_1}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(1)}}\cdots Y^{i_n}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(n)}}\,\delta^{i_{n+1}}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(n+1)}}\Big\rangle \qquad (3.1)\end{aligned}$$

Now Wick contract with (1.1), permuting with τ for all the possible combinations

$$\begin{aligned}&\frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma,\tau\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2) \\ &\quad \delta^{j_{\tau(1)}}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(1)}}\ \delta^{i_1}_{j_{\alpha_2\tau(1)}}\ \cdots\ \delta^{j_{\tau(n)}}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(n)}}\ \delta^{i_n}_{j_{\alpha_2\tau(n)}}\ \delta^{i_{n+1}}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2(n+1)}} \\ &= \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma,\tau\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2)\delta^{i_1}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2\tau^{-1}\alpha_2\tau(1)}}\cdots\delta^{i_{n+1}}_{i_{\rho_1\sigma^{-1}\alpha_1\sigma\rho_2\tau^{-1}\alpha_2\tau(n+1)}} \\ &= \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma,\tau\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2)\ \mathrm{tr}(\rho_1\,\sigma^{-1}\alpha_1\sigma\ \rho_2\ \tau^{-1}\alpha_2\tau\ \mathbb{I}_N^{n+1}) \qquad (3.2)\end{aligned}$$

See figure 5 for the diagrammatic representation of this trace. We can expand it in characters of S_{n+1} and dimensions of $\mathrm{U}(N)$ $(n+1)$-box representations

$$\begin{aligned}&\Big\langle \mathrm{tr}(\alpha_2\, X^{\dagger\mu}Y^{\dagger\nu}) : \mathrm{tr}([X,Y][X^\dagger,Y^\dagger]) : \mathrm{tr}(\alpha_1\, X^\mu Y^\nu)\Big\rangle \\ &\quad = \frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\sum_{\sigma,\tau\in S_\mu\times S_\nu}\sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2)\sum_{T\vdash n+1}\chi_T(\rho_1\,\sigma^{-1}\alpha_1\sigma\ \rho_2\ \tau^{-1}\alpha_2\tau)\mathrm{Dim}T \quad (3.3)\end{aligned}$$

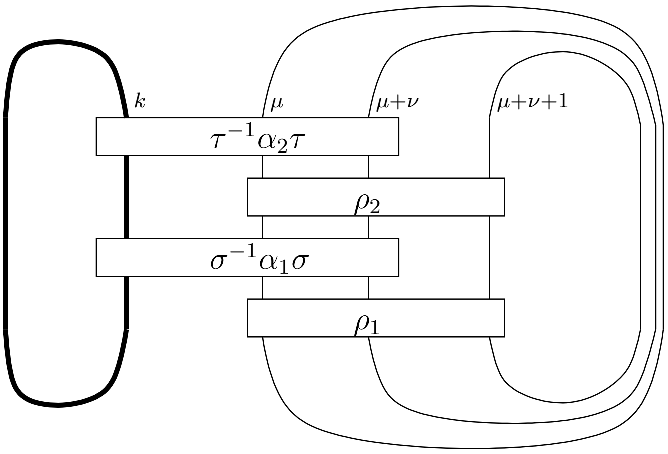


Figure 5: One-loop correlator

4. Operator mixing

Operator mixing between single- and multi-trace operators at one-loop has been well studied (see for example [14 − 17, 6]). Here we will consider the mixing of a different basis of operators.

In [1] a complete basis of gauge-invariant operators was constructed that diagonalises the tree-level correlator for a theory with $\mathrm{U}(M)$ global flavour symmetry and $\mathrm{U}(N)$ gauge symmetry. This Clebsch-Gordan basis tells us how to mesh the $\mathrm{U}(2)$ (or more generally the $\mathrm{U}(M)$) representation, which dictates how the operator transforms under the flavour group, with the $\mathrm{U}(N)$ representation, which controls the multi-trace structure

$$\begin{aligned}\mathcal{O}[\Lambda,\mu,\nu,\beta;R;\tau] &\equiv \frac{1}{(n!)^2}\sum_{\alpha,\sigma\in S_n} B_{j\beta}\, S^{\tau,\Lambda}{}_{i}{}^{R}{}_{k}{}^{R}{}_{l}\; D^{\Lambda}_{ij}(\sigma)D^{R}_{kl}(\alpha)\,\mathrm{tr}(\alpha\sigma\; X^{\mu}Y^{\nu}\;\sigma^{-1})\\ &= \frac{1}{n!}\sum_{\alpha\in S_n} B_{j\beta}\, S^{\tau,\Lambda}{}_{j}{}^{R}{}_{p}{}^{R}{}_{q}\; D^{R}_{pq}(\alpha)\,\mathrm{tr}(\alpha\; X^{\mu}Y^{\nu}) \end{aligned}\qquad (4.1)$$

The equality follows from identity (A.2). Here Λ labels the $\mathrm{U}(2)$ representation and $[\mu,\nu,\beta]$ labels the state within Λ: μ,ν label the number of fields X,Y and $\beta \in \{1,\dots g(\overbrace{\square\square\cdots\square}^{\mu},\overbrace{\square\square\cdots\square}^{\nu};\Lambda)\}$ labels the semistandard tableau with field content X^μ and Y^ν.[5] R labels the $\mathrm{U}(N)$ representation, which dictatess the multi-trace structure of the operator. τ labels the number of times Λ appears in the symmetric group tensor product $R\otimes R$ (also called the inner product). $S^{\tau,\Lambda}{}_{j}{}^{R}{}_{p}{}^{R}{}_{q}$ is the S_n Clebsch-Gordan coefficient for this tensor

[5]The Littlewood-Richardson coefficient g counts the number of times Λ appears in $\overbrace{\square\square\cdots\square}^{\mu}\circ\overbrace{\square\square\cdots\square}^{\nu}$, where $\circ$ is the tensor product for $\mathrm{U}(2)$ and the outer product for the symmetric group S_n. For such tensor products of totally symmetric representations, this Littlewood-Richardson coefficient is also known as the Kostka number for Λ and field content μ,ν. In the $\mathrm{U}(2)$ case this is all a bit trivial because $g(\square\square\cdots\square,\square\square\cdots\square;\Lambda)$ is either zero or one, but the β multiplicity becomes non-trivial for $\mathrm{U}(M)$ with $M\geq 3$. $B_{j\beta}$ is the branching coefficient for the restriction of Λ to the representation $\overbrace{\square\square\cdots\square}^{\mu}\circ\overbrace{\square\square\cdots\square}^{\nu}$ of its $S_\mu\times S_\nu$ subgroup.

product.[6] From the unitary group perspective S blends the global symmetry $\mathrm{U}(2)$ with the gauge symmetry $\mathrm{U}(N)$. $D^R_{pq}(\alpha)$ is the real orthogonal Young-Yamanouchi $d_R \times d_R$ matrix for the representation R of the symmetry group S_n. It is constructed in section 7 of Hamermesh [18] following the presentation by Yamanouchi [19]. All of these factors are explained in detail in [1].

At tree level these operators are diagonal

$$\left\langle \mathcal{O}^\dagger[\Lambda_2, \mu_2, \nu_2, \beta_2; R_2; \tau_2]\ \mathcal{O}[\Lambda_1, \mu_1, \nu_1, \beta_1; R_1; \tau_1] \right\rangle = \delta^{[\Lambda_1,\mu_1,\nu_1,\beta_1;R_1;\tau_1]}_{[\Lambda_2,\mu_2,\nu_2,\beta_2;R_2;\tau_2]} \frac{\mu_1!\nu_1!\ \mathrm{Dim}R_1}{d^2_{R_1}} \tag{4.2}$$

Now consider the one-loop correlator

$$\left\langle \mathcal{O}^\dagger[\Lambda_2, \mu, \nu, \beta_2; R_2; \tau_2] : \mathrm{tr}([X,Y][X^\dagger, Y^\dagger]) : \mathcal{O}[\Lambda_1, \mu, \nu, \beta_1; R_1; \tau_1] \right\rangle \tag{4.3}$$

A priori we know that the one-loop dilatation operator will not mix the $\mathrm{U}(2)$ representations labelled by Λ and the states within those representations labelled by $[\mu, \nu, \beta]$ because the one-loop dilatation operator commutes with the classical generators of $\mathrm{U}(2)$ (and indeed of the full classical superconformal group [20]).[7] There is however no reason why the $\mathrm{U}(N)$ representations R controlling the multi-trace structure shouldn't mix and we will now analyse this using our one-loop result (3.3).

The first thing we notice, following techniques from [1], is that for a general function of a permutation $f(\alpha)$

$$\frac{1}{n!} \sum_{\alpha \in S_n} B_{j\beta} S^{\tau,\Lambda R R}_{\ \ j\ p\ q} D^R_{pq}(\alpha) \sum_{\sigma \in S_\mu \times S_\nu} f(\sigma^{-1}\alpha\sigma) = \frac{\mu!\nu!}{n!} \sum_{\alpha \in S_n} B_{j\beta} S^{\tau,\Lambda R R}_{\ \ j\ p\ q} D^R_{pq}(\alpha) f(\alpha) \tag{4.4}$$

so that for the one-loop correlator (3.3) we can absorb the $S_\mu \times S_\nu$ sums.[8]

Thus if we concentrate on the $\mathrm{U}(N)$ representation parts of equations (3.3) and (4.3) we find

$$\sum_{\alpha_1,\alpha_2 \in S_n} D^{R_1}_{p_1q_1}(\alpha_1) D^{R_2}_{p_2q_2}(\alpha_2) \sum_{T \vdash n+1} \chi_T(\rho_1\ \alpha_1\ \rho_2\ \alpha_2) \mathrm{Dim}T \tag{4.5}$$

If we expand the character, which is just a trace of S_{n+1} representing matrices for T, we get

$$\sum_{\alpha_1,\alpha_2 \in S_n} D^{R_1}_{p_1q_1}(\alpha_1) D^{R_2}_{p_2q_2}(\alpha_2) \sum_{T \vdash n+1} D^T_{ab}(\rho_1) D^T_{bc}(\alpha_1) D^T_{cd}(\rho_2) D^T_{da}(\alpha_2) \mathrm{Dim}T \tag{4.6}$$

We can pick out the sum over α_1 say

$$\sum_{\alpha_1 \in S_n} D^{R_1 \vdash n}_{p_1q_1}(\alpha_1) D^{T \vdash n+1}_{bc}(\alpha_1) \tag{4.7}$$

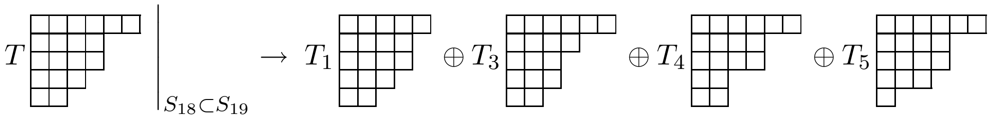
---

[6] $S^{\tau,\Lambda\ R\ R}_{\ \ j\ \ p\ \ q}$ for S_n is exactly analogous to the $3j$-symbol for SU(2), which is just an expression of the Clebsch-Gordan coefficients.

[7] We thank Sanjaye Ramgoolam for discussions on this point.

[8] Another way of understanding this is that $\alpha \mapsto \sigma^{-1}\alpha\sigma$ for $\sigma \in S_\mu \times S_\nu$ is a symmetry of the operator $\mathrm{tr}(\alpha\ X^\mu Y^\nu)$.

α_1 is in the S_n subgroup of S_{n+1}. As a representation of S_n the representation T is reducible. It reduces to those n-box representations of S_n whose Young diagrams differ by a box from T. Consider the example used in section 7 of Hamermesh [18]



(4.8)

The index r of T_r labels the row from which the box was removed from T. This direct product structure is manifest for the representation matrices constructed by Young and Yamanouchi, where the matrix D^T is block-diagonal for elements of the subgroup $\sigma \in S_n \subset S_{n+1}$. For example (4.8)

$$D^{T\,\vdash\, n+1}(\sigma) = \begin{pmatrix} D^{T_1\,\vdash\, n}(\sigma) & & & \\ & D^{T_3\,\vdash\, n}(\sigma) & & \\ & & D^{T_4\,\vdash\, n}(\sigma) & \\ & & & D^{T_5\,\vdash\, n}(\sigma) \end{pmatrix} \tag{4.9}$$

For a representation T_r of S_n we can then apply the identity

$$\sum_{\alpha_1 \in S_n} D^{R_1\,\vdash\, n}_{p_1 q_1}(\alpha_1) D^{T_r\,\vdash\, n}_{bc}(\alpha_1) = \frac{n!}{d_{T_r}} \delta^{R_1 T_r} \delta_{p_1 b} \delta_{q_1 c} \tag{4.10}$$

This identity follows from Schur's lemma and the orthogonality of the representing matrices.

Given the block-diagonal decomposition of D^T on α_1 and α_2 we find that (4.6) is only non-zero if $R_1 = T_r$ and $R_2 = T_s$ for some T and for some r and s labelling the row from which a box is removed from T. If there is no T such that we can remove a single box to reach R_1 and R_2 then the one-loop correlator vanishes. This is the crucial point.

If $R_1 \neq R_2$ then there is at most one representation T of S_{n+1} satisfying this property and we find that (4.6) becomes

$$\frac{n!}{d_{T_r}} \frac{n!}{d_{T_s}} D^T_{\underset{s}{q_2}\underset{r}{p_1}}(\rho_1) D^T_{\underset{r}{q_1}\underset{s}{p_2}}(\rho_2) \mathrm{Dim} T \tag{4.11}$$

The letters underneath the matrix indices indicate the sub-range of the d_T indices of D^T over which the index ranges. For example, here q_2 only ranges over the d_{T_s} indices of D^T in the appropriate s sub-row of D^T and p_1 only ranges over the d_{T_r} indices in the r sub-column (see for example the matrix in (4.9)).[9] Thus for $D^T_{\underset{s}{q_2}\underset{r}{p_1}}(\rho_1)$ q_2 and p_1 label elements in an off-diagonal sub-block of D^T. This does not vanish because ρ_1 is a generic element of S_{n+1} not in its S_n subgroup.

So if there exists a T for which $R_1 = T_r$ and $R_2 = T_s$ and $R_1 \neq R_2$

$$\begin{aligned} &\Big\langle \mathcal{O}^\dagger[\Lambda_2, \mu, \nu, \beta_2; T_s; \tau_2] : \mathrm{tr}([X,Y][X^\dagger, Y^\dagger]) : \mathcal{O}[\Lambda_1, \mu, \nu, \beta_1; T_r; \tau_1] \Big\rangle \\ &= \frac{\mu\nu\mu!\nu!}{d_{T_r} d_{T_s}} B_{j_1\beta_1} S^{\tau_1,\Lambda_1\, T_r\, T_r}_{\;\;j_1\;\; p_1\, q_1} B_{j_2\beta_2} S^{\tau_2,\Lambda_2\, T_s\, T_s}_{\;\;j_2\;\; p_2\, q_2} \sum_{\rho_1,\rho_2 \in S_{n+1}} h(\rho_1,\rho_2) D^T_{\underset{s}{q_2}\underset{r}{p_1}}(\rho_1) D^T_{\underset{r}{q_1}\underset{s}{p_2}}(\rho_2) \mathrm{Dim} T \end{aligned} \tag{4.12}$$

[9]To be more sophisticated, s is the first number in the Yamanouchi symbol for the index of T and q_2 is the rest of the symbol for T_s.

If we use the more symmetric expression for h in (2.13) then we can use identity (A.2) from appendix A to get

$$
\begin{aligned}
-\frac{\mu\nu\mu!\nu!}{d_{T_r} d_{T_s}} B_{j_1\beta_1}\; S^{\tau_1,\Lambda_1}{}_{k_1}{}^{T_r}_{p_1}{}^{T_r}_{q_1}\; B_{j_2\beta_2}\; S^{\tau_2,\Lambda_2}{}_{k_2}{}^{T_s}_{p_2}{}^{T_s}_{q_2} \\
D^{\Lambda_1}_{j_1k_1}(1-(\mu,n))\; D^{\Lambda_2}_{j_2k_2}(1-(\mu,n))\; D^{T}_{\substack{q_2p_1\\ s\ r}}((\mu,n+1))D^{T}_{\substack{q_1p_2\\ r\ s}}((n,n+1))\mathrm{Dim}T
\end{aligned}
\tag{4.13}
$$

This expression nicely encodes the vanishing of the one-loop correlator for the half-BPS operators transforming in the symmetric representation of the flavour group (for $\Lambda = \square\square\cdots\square$, $D^\Lambda(\sigma)=1\ \forall\sigma$).

Some hints on how to simplify this expression further, and how one might extract explicitly the orthogonality of U(2) representations, is given in appendix D.

If $R_1 = R_2 \equiv R$ then we must sum over all the representations T of S_{n+1} with $T_r = R$

$$
\begin{aligned}
&\left\langle \mathcal{O}^\dagger[\Lambda_2,\mu,\nu,\beta_2;R;\tau_2] : \mathrm{tr}([X,Y][X^\dagger,Y^\dagger]) : \mathcal{O}[\Lambda_1,\mu,\nu,\beta_1;R;\tau_1] \right\rangle \\
&= \sum_{T \text{ s.t. } R=T_r} \frac{\mu\nu\mu!\nu!}{d_{T_r}^2} B_{j_1\beta_1}\; S^{\tau_1,\Lambda_1}{}_{j_1}{}^{T_r}_{p_1}{}^{T_r}_{q_1}\; B_{j_2\beta_2}\; S^{\tau_2,\Lambda_2}{}_{j_2}{}^{T_r}_{p_2}{}^{T_r}_{q_2} \sum_{\rho_1,\rho_2\in S_{n+1}} h(\rho_1,\rho_2) D^{T}_{\substack{q_2p_1\\ r\ r}}(\rho_1) D^{T}_{\substack{q_1p_2\\ r\ r}}(\rho_2)\mathrm{Dim}T
\end{aligned}
$$

An example of these mixing properties is worked out for $\Lambda = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}$ in appendix E.

Some general comments:

- We can interpret the U(N) representation $T \vdash n+1$ as an intermediate channel through which the operators mix via the 'overlapping' of $R_1 \vdash n$ and $R_2 \vdash n$ with T.
- Given that smaller Young diagrams are more likely to be related to each other by moving a box than larger diagrams, mixing at one loop is much more likely for smaller representations than larger ones. Larger ones can be considered practically diagonal at 1-loop (but not at higher loops, see section 5).

4.1 Dilatation operator

We can now apply this analysis to the one-loop dilatation operator.

$$
\Delta^{(1)}\ \mathcal{O}[\Lambda,\mu,\nu,\beta;R;\tau] = \sum_{S,\tau'} C^{R,\tau}_{S,\tau'}\ \mathcal{O}[\Lambda,\mu,\nu,\beta;S;\tau'] \tag{4.14}
$$

S must be obtainable by removing a box from R and then putting it back somewhere. We can obtain the matrix $C^{R,\tau}_{S,\tau'}$ by reverse-engineering the one-loop mixing (4.13) using the tree-diagonality of the Clebsch-Gordan basis (4.2). We can see for example that for $R \neq S$ which mix via $T \vdash n+1$ we can factor out the N dependence

$$
\begin{aligned}
C^{R;\tau}_{S;\tau'} &= -\,\mu\nu \frac{d_S}{d_R}\frac{\mathrm{Dim}T}{\mathrm{Dim}S} B_{j_1\beta}\; S^{\tau,\Lambda}{}_{k_1}{}^{R}_{p_1}{}^{R}_{q_1}\; B_{j_2\beta}\; S^{\tau',\Lambda}{}_{k_2}{}^{S}_{p_2}{}^{S}_{q_2} \\
&\qquad D^{\Lambda}_{j_1k_1}(1-(\mu,n))\; D^{\Lambda}_{j_2k_2}(1-(\mu,n))\; D^{T}_{\substack{q_2p_1\\ s\ r}}((\mu,n+1))D^{T}_{\substack{q_1p_2\\ r\ s}}((n,n+1)) \\
&\propto \frac{\mathrm{Dim}T}{\mathrm{Dim}S} \;\propto\; N-i+j
\end{aligned}
\tag{4.15}
$$

where i labels the row coordinate and j the column coordinate of the box R has that S doesn't (see equation (A.1)).

The kernel of this map provides the $\frac{1}{4}$-BPS operators [21, 22], but we have no further insight on how to obtain a pleasing group theoretic expression for these operators beyond the hints given in [1] concerning the dual basis [23, 24]. Something like the dual basis seems particularly relevant given that it arose in the SU(N) context [25, 23] from knocking boxes off representations to differentiate Schur polynomials.

5. Higher loops and other sectors

If we assume that higher ℓ-loop contributions to the correlator can always be written in terms of an effective vertex like (1.4) (it works for two loops [11]) then we guess that they can be written in terms of $S_{n+\ell}$ and $\mathrm{U}(N)$ group theory

$$\sum_{\sigma,\tau\in S_\mu\times S_\nu}\ \sum_{\rho_1,\rho_2\in S_{n+\ell}} h_\ell(\rho_1,\rho_2) \sum_{T\vdash n+\ell} \chi_T(\rho_1\ \sigma^{-1}\alpha_1\sigma\ \rho_2\ \tau^{-1}\alpha_2\tau)\mathrm{Dim}T \tag{5.1}$$

$h_\ell(\rho_1,\rho_2)$ only takes non-zero values on a few permutations of $\ell+1$ of the $\{1,\dots n\}$ indices (where the derivative acts) and the $n+1,\ \dots n+\ell$ indices. The σ and τ construction permutes the X's and Y's for the product rule.

This guess is informed by the leading planar $N^{n+\ell}$ contribution to the ℓ-loop term, which is provided by the large N behaviour of Dim$T$ when T has $n+\ell$ boxes (see equation (A.1)).

As a consequence of this structure $\mathcal{O}[\Lambda_1,\mu,\nu,\beta_1;R_1;\tau_1]$ and $\mathcal{O}[\Lambda_2,\mu,\nu,\beta_2;R_2;\tau_2]$ can only mix at ℓ loops if we can reach the same $(n+\ell)$-box Young diagram T by adding ℓ boxes to each of the $\mathrm{U}(N)$ representations R_1 and R_2.

An alternative way of saying this is that if two $\mathrm{U}(N)$ representations R_1 and R_2 have k boxes in the same position then they can first mix at $n-k$ loops, since we have enough boxes to add to R_1 to reproduce the shape of R_2.

This means that all operators of length n can mix at $n-1$ loops, because all diagrams share the first box in the upper lefthand corner.

We have focused here on the $\mathrm{U}(2)\subset\mathrm{SU}(4)\subset\mathrm{PSU}(2,2|4)$ sector of the full symmetry group of $\mathcal{N}=4$. It seems fairly obvious that this work extends to $\mathrm{U}(3)$ because the effective vertex gains similar terms to the $\mathrm{U}(2)$ vertex and the basis of [1] accommodates a general $\mathrm{U}(M)$ flavour symmetry; the remaining sectors [20] would require more work, especially given that the basis constructed in [1] doesn't extend there yet. It would be particularly interesting to extend the work of [26] and understand the counting of sixteenth-BPS operators at one loop in the non-planar limit, and hence gain an understanding of black hole entropy via AdS/CFT.

There are satisfying group-theoretic expressions for extremal higher-point correlators of the Clebsch-Gordan operators at tree level [1]. It would be interesting to see how much of this structure survives at one loop.

Finally we point out that another complete basis in the $\mathrm{U}(2)$ sector, the restricted Schur polynomials, have neat tree-level two-point functions and their one-loop properties have been studied [27 – 30].

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6. Discussion

The main motivation for studying these operators and their mixing is that $\mathcal{N}=4$ super Yang-Mills has a dual string theory on an $AdS_5 \times S^5$ background [31 – 33]. We give here some techniques that allow us better control of the regime where the length of operators is arbitrary, λ is non-trivial and N is finite, the regime where the 'strong' Maldacena conjecture might hold beyond the planar 't Hooft limit.

We have no clear idea what the tree-diagonal operators constructed in [1] correspond to on the string theory side. They are not eigenstates of the one-loop dilatation operator, but their limited mixing might pave the way for such a diagonalisation. The BPS operators map to giant graviton branes when the operators are large [34 – 37]. Some hints on how to obtain these operators from the Clebsch-Gordan basis were given in [1].

On the string side splitting of strings is suppressed by $g_s \sim 1/N$. One lesson perhaps is that it is fruitful to think in terms of Young diagrams gaining and losing boxes as well as in terms of traces splitting and joining. An advantage of the Young diagram methods is that the finite N constraint is clear in terms of a limit on the number of rows. It would be interesting to understand how this constraint [38] is implemented for general string states, particularly given that it is reminiscent of the level cutoff of Wess-Zumino-Witten models [39].

Representation theory and Schur-Weyl duality played an important part in our understanding of 2d Yang-Mills and its string dual [40 – 42]. We hope that Schur-Weyl duality, and the interplay between the gauge group and the symmetry group, will provide vital clues for our understanding of $d=4, \mathcal{N}=4$ supersymmetric Yang-Mills and the string on $AdS_5 \times S^5$.

Acknowledgments

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A. Conventions and formulae

$R \vdash n$ is an irreducible representation of S_n and also of $\mathrm{U}(N)$. It can be drawn as a Young diagram with n boxes; representations of $\mathrm{U}(N)$ have at most N rows.

$d_R = \frac{n!}{\prod_{i,j} h_{i,j}}$ is the dimension of the symmetric group representation R, where $h_{i,j}$ is the hook length for the box in the ith row and jth column.

$\mathrm{Dim}R$ is the dimension of the unitary group $\mathrm{U}(N)$ representation R, given by

$$\mathrm{Dim}R = \prod_{(i,j)\in R} \frac{N-i+j}{h_{i,j}} \tag{A.1}$$

Again i labels the row coordinate and j the column coordinate of each box in R.

The S_n Clebsch-Gordan coefficients satisfy for a permutation $\sigma \in S_n$

$$\sum_{j,l} D^S_{ij}(\sigma) D^T_{kl}(\sigma)\; S^{\tau_R,\,R\ S\ T}_{\quad\ s\ j\ l} = \sum_t D^R_{ts}(\sigma)\; S^{\tau_R,\,R\ S\ T}_{\quad\ t\ i\ k} \tag{A.2}$$

$$\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3} =$$

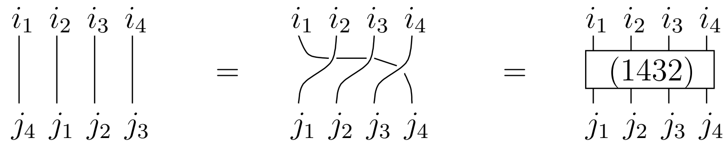


Figure 6: From delta functions to diagrams to permutations

$$\delta^{i_k}_{j_{\beta\alpha(k)}} =$$

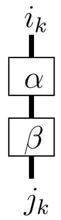


Figure 7: Permutations in series; thick lines represent many strands

$$\delta^{i_{\beta\alpha(k)}}_{j_k} = \delta^{i_k}_{j_{\alpha^{-1}\beta^{-1}(k)}} =$$

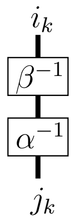


Figure 8: Permutations on the upper index

This tells us how to obtain matrix elements from the symmetric group inner product $R \in S \otimes T$. τ_R labels the multiplicity of R in $S \otimes T$.

B. Diagrammatics

Diagrammatics [4] encode the 't Hooft double-line indices. We follow the index lines with delta functions and permutations, see for example figure 6. We read the permutations in the diagrams from the top down. This is also illustrated in figure 7, where we remember that in the permutation $\beta\alpha$ we read from right to left, so that α acts first followed by β. Also in figure 7 we clump several strands labelled by k into a single thick strand, for clarity.

If we write down a series of delta functions we can always alter the order in which we write them down with any $\sigma \in S_n$, given that they are just numbers

$$\delta^{i_1}_{j_{\alpha(1)}} \cdots \delta^{i_n}_{j_{\alpha(n)}} = \delta^{i_{\sigma(1)}}_{j_{\alpha\sigma(1)}} \cdots \delta^{i_{\sigma(n)}}_{j_{\alpha\sigma(n)}} \tag{B.1}$$

This allows us to deal with permutations on the upper index, see figure 8.

If we have $\delta^{i_{\alpha(k)}}_{j_{\beta(k)}}$ and we set $j_k = i_{\sigma(k)}$ then we get

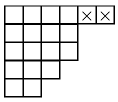
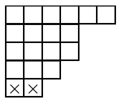
$$\delta^{i_{\alpha(k)}}_{j_{\beta(k)}} \delta^{j_k}_{i_{\sigma(k)}} = \delta^{i_{\alpha\beta^{-1}(k)}}_{j_k} \delta^{j_k}_{i_{\sigma(k)}} = \delta^{i_{\alpha\beta^{-1}(k)}}_{i_{\sigma(k)}} = \delta^{i_{\alpha(k)}}_{i_{\sigma\beta(k)}} \tag{B.2}$$

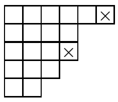
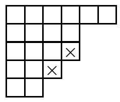
C. Symmetric group representation matrices

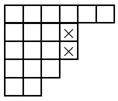
Here we briefly review the Young-Yamanouchi construction of real orthogonal representing matrices for an S_n representation T [19], which is summarised in Hamermesh [18].

The matrices are constructed recursively: we assume that we know all the representation matrices for all the representations of S_k for $k < n$. We also know that on elements of the subgroup $S_{n-1} \subset S_n$ the representation T reduces to a sum of those irreducible representations of S_{n-1} that have one box removed from T (see for example equations (4.8) and (4.9)). Given that we know all the representation matrices for all of S_{n-1} we know the form of the representation matrices for T on $S_{n-1} \subset S_n$.

To reach those permutations that also act on the last object, all we need to know in addition is the matrix for $(n-1, n)$, $D^T((n-1,n))$. To obtain this, we observe that this matrix commutes with all the matrices for the subgroup $S_{n-2} \subset S_n$, since they are permuting separate groups of objects. We can then use Schur's lemmas to obtain $D^T((n-1,n))$.

Type I: T_{11}  and T_{55} 

Type II: $T_{13} = T_{31}$  , $T_{34} = T_{43}$  , $\cdots$

Type III: T_{32}  (C.1)

To get the representing matrices of T on $S_{n-2} \subset S_n$, we must reduce T by knocking off two boxes. We label these irreps of S_{n-2} by T_{rs} where r is the row from which the first box is knocked, s the second. There are three different situations when we knock off two boxes, called Type I, II and III. These are exhibited for the example given in equation (4.8).

For Type I and Type III the second box can only be knocked off *after* the first one: Type I is when the second box is to the left of the first on the same row; Type III is when the second box is above the first on the same column. For Type II both boxes can be knocked off independently and $T_{rs} = T_{sr}$.

This reduction of S_n representations on subgroups is also called *branching*.

D. Further analysis of the matrices

Here we analyse in more detail the one-loop mixing of the Clebsch-Gordan basis for $R_1 = T_r$ and $R_2 = T_s$ and $r \neq s$ given in (4.13).

It turns out, given the recursive construction of the representing matrices (see appendix C), that we know $D^T_{\underset{r}{q_1}\underset{s}{p_2}}((n, n+1))$ exactly. If we further restrict T to S_{n-1} then the representation reduces to Young diagrams with two boxes removed from T. $T_{rs} = T_{sr}$ is the common S_{n-1} Young diagram obtained when boxes are removed both from the rth and sth rows (see figure 9). It is Type II because the boxes can be removed independently. Because

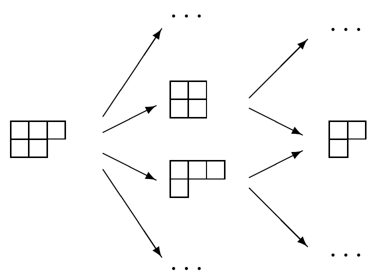


Figure 9: Restriction pattern for $S_{n+1} \to S_n \to S_{n-1}$

$(n, n+1)$ commutes with all elements of S_{n-1}, $D^T_{\substack{q_1\\r}\substack{p_2\\s}}((n,n+1))$ is only non-zero in the case

$$D^T_{\substack{q_1\\rs}\substack{p_2\\sr}}((n,n+1)) = \frac{\sqrt{\tau^2_{rs,rs} - 1}}{|\tau_{rs,rs}|} E_{rs,sr} \tag{D.1}$$

where $E_{rs,sr}$ is the identity matrix. If the row lengths of T are given by t_r then $\tau_{rs,rs}$ is[10]

$$\tau_{rs,rs} = (t_r - r) - (t_s - s) \tag{D.2}$$

Unfortunately we can't work the same magic on $D^T_{\substack{q_2\\s}\substack{p_1\\r}}((\mu, n+1))$.

There are also branching-type recursive relations for the Clebsch-Gordan coefficients (see the end of section 7 of Hamermesh [18]).

Given that we know (4.13) is diagonal in the U(2) states, this may imply non-trivial identities for these symmetric group reduction formulae.

E. Example

We consider the case with U(2) representation $\Lambda = \yng(2,2)$ and field content $XXYY$. This must be a highest weight state of Λ because the field content matches the rows of Λ. Thus β is unique.

The three allowed $\mathrm{U}(N)$ representations are $R = \yng(2,2), \yng(3,1), \yng(2,1,1)$, for which Λ only appears once in the symmetric group inner product $R \otimes R$.

Here $\Phi_r \Phi^r = \epsilon_{rs} \Phi^r \Phi^s = [X, Y]$.

$$\mathcal{O}\left[\Lambda = \yng(2,2); R = \yng(2,2)\right] = \frac{1}{12\sqrt{2}} [\mathrm{tr}(\Phi_r \Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) + \mathrm{tr}(\Phi_r \Phi^r \Phi_s \Phi^s)] \tag{E.1}$$

$$\mathcal{O}\left[\Lambda = \yng(2,2); R = \yng(3,1)\right] = \frac{1}{12\sqrt{6}} [\mathrm{tr}(\Phi_r \Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) + \mathrm{tr}(\Phi_r \Phi_s)\,\mathrm{tr}(\Phi^r \Phi^s) - \mathrm{tr}(\Phi_r \Phi^r \Phi_s \Phi^s)] \tag{E.2}$$

$$\mathcal{O}\left[\Lambda = \yng(2,2); R = \yng(2,1,1)\right] = \frac{1}{12\sqrt{6}} [\mathrm{tr}(\Phi_r \Phi_s)\,\mathrm{tr}(\Phi^r)\,\mathrm{tr}(\Phi^s) - \mathrm{tr}(\Phi_r \Phi_s)\,\mathrm{tr}(\Phi^r \Phi^s) - \mathrm{tr}(\Phi_r \Phi^r \Phi_s \Phi^s)] \tag{E.3}$$

[10] $\tau_{rs,rs}$ is also known as the *axial distance*.

The tree level correlator is diagonal

$$\begin{pmatrix} \frac{1}{12}N^2(N^2-1) & & \\ & \frac{1}{18}N(N^2-1)(N+2) & \\ & & \frac{1}{18}N(N^2-1)(N-2) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Dim}\,\ydiagram{2,2} & & \\ & \frac{4}{9}\mathrm{Dim}\,\ydiagram{3,1} & \\ & & \frac{4}{9}\mathrm{Dim}\,\ydiagram{2,1,1} \end{pmatrix} \tag{E.4}$$

At one loop everything mixes

$$\begin{pmatrix} \frac{1}{4}N^3(1-N^2) & \frac{1}{4\sqrt{3}}N^2(N^2-1)(N+2) & \frac{1}{4\sqrt{3}}N^2(N^2-1)(N-2) \\ \frac{1}{4\sqrt{3}}N^2(N^2-1)(N+2) & \frac{1}{12}N(1-N^2)(N+2)^2 & \frac{1}{12}N(1-N^2)(N^2-4) \\ \frac{1}{4\sqrt{3}}N^2(N^2-1)(N-2) & \frac{1}{12}N(1-N^2)(N^2-4) & \frac{1}{12}N(1-N^2)(N-2)^2 \end{pmatrix}$$

$$= \begin{pmatrix} -3N\mathrm{Dim}\,\ydiagram{2,2} & 2\sqrt{3}\mathrm{Dim}\,\ydiagram{3,2} & 2\sqrt{3}\mathrm{Dim}\,\ydiagram{2,2,1} \\ 2\sqrt{3}\mathrm{Dim}\,\ydiagram{3,2} & -\frac{2}{3}(N+2)\mathrm{Dim}\,\ydiagram{3,1} & -\frac{5}{3}\mathrm{Dim}\,\ydiagram{3,1,1} \\ 2\sqrt{3}\mathrm{Dim}\,\ydiagram{2,2,1} & -\frac{5}{3}\mathrm{Dim}\,\ydiagram{3,1,1} & -\frac{2}{3}(N-2)\mathrm{Dim}\,\ydiagram{2,1,1} \end{pmatrix} \tag{E.5}$$

The diagonal terms seem to be the dimension of the irrep. enhanced by the contribution for a specific box, furthest from the top left.

F. Code

All correlators at tree level and one loop can be checked with the correlator program written in python and released under the GNU General Public Licence at http://www.nworbmot.org/code/.

References

[1] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal multi-matrix correlators and BPS operators in $N = 4$ SYM*, *JHEP* **02** (2008) 030 [`arXiv:0711.0176`].

[2] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[3] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [`hep-th/0111222`].

[4] S. Corley and S. Ramgoolam, *Finite factorization equations and sum rules for BPS correlators in $N = 4$ SYM theory*, *Nucl. Phys.* **B 641** (2002) 131 [`hep-th/0205221`].

[5] N.R. Constable et al., *PP-wave string interactions from perturbative Yang-Mills theory*, *JHEP* **07** (2002) 017 [hep-th/0205089].

[6] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 650** (2003) 125 [hep-th/0208178].

[7] D.J. Gross, A. Mikhailov and R. Roiban, *A calculation of the plane wave string Hamiltonian from $N = 4$ super-Yang-Mills theory*, *JHEP* **05** (2003) 025 [hep-th/0208231].

[8] R.A. Janik, *BMN operators and string field theory*, *Phys. Lett.* **B 549** (2002) 237 [hep-th/0209263].

[9] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett.* **B 558** (2003) 229 [hep-th/0212269].

[10] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[11] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[12] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[13] S. Bellucci, P.Y. Casteill, J.F. Morales and C. Sochichiu, *Spin bit models from non-planar $N = 4$ SYM*, *Nucl. Phys.* **B 699** (2004) 151 [hep-th/0404066].

[14] G. Arutyunov and S. Frolov, *On the correspondence between gravity fields and CFT operators*, *JHEP* **04** (2000) 017 [hep-th/0003038].

[15] G. Arutyunov, S. Penati, A.C. Petkou, A. Santambrogio and E. Sokatchev, *Non-protected operators in $N = 4$ SYM and multiparticle states of AdS_5 SUGRA*, *Nucl. Phys.* **B 643** (2002) 49 [hep-th/0206020].

[16] M. Bianchi, B. Eden, G. Rossi and Y.S. Stanev, *On operator mixing in $N = 4$ SYM*, *Nucl. Phys.* **B 646** (2002) 69 [hep-th/0205321].

[17] N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, *Operator mixing and the BMN correspondence*, *JHEP* **10** (2002) 068 [hep-th/0209002].

[18] M. Hamermesh, *Group theory and its application to physical problems*, Addison-Wesley Publishing Company, U.S.A. (1962).

[19] T. Yamanouchi, Proc. Phys. Math. Soc. Japan **19** (1937) 436.

[20] N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015].

[21] A.V. Ryzhov, *Quarter BPS operators in $N = 4$ SYM*, *JHEP* **11** (2001) 046 [hep-th/0109064].

[22] E. D'Hoker, P. Heslop, P. Howe and A.V. Ryzhov, *Systematics of quarter BPS operators in $N = 4$ SYM*, *JHEP* **04** (2003) 038 [hep-th/0301104].

[23] T.W. Brown, *Half-BPS* $\mathrm{SU}(N)$ *correlators in $N = 4$ SYM*, hep-th/0703202.

[24] T. Brown, R. de Mello Koch, S. Ramgoolam and N. Toumbas, *Correlators, probabilities and topologies in $N = 4$ SYM*, *JHEP* **03** (2007) 072 [hep-th/0611290].

[25] R. de Mello Koch and R. Gwyn, *Giant graviton correlators from dual* SU(N) *super Yang-Mills theory*, *JHEP* **11** (2004) 081 [hep-th/0410236].

[26] R.A. Janik and M. Trzetrzelewski, *Supergravitons from one loop perturbative $N = 4$ SYM*, *Phys. Rev.* **D 77** (2008) 085024 [arXiv:0712.2714].

[27] R. de Mello Koch, J. Smolic and M. Smolic, *Giant gravitons — with strings attached (I)*, *JHEP* **06** (2007) 074 [hep-th/0701066].

[28] R. de Mello Koch, J. Smolic and M. Smolic, *Giant gravitons — with strings attached (II)*, *JHEP* **09** (2007) 049 [hep-th/0701067].

[29] D. Bekker, R. de Mello Koch and M. Stephanou, *Giant gravitons — with strings attached (III)*, *JHEP* **02** (2008) 029 [arXiv:0710.5372].

[30] R. Bhattacharyya, S. Collins and R.d.M. Koch, *Exact multi-matrix correlators*, *JHEP* **03** (2008) 044 [arXiv:0801.2061].

[31] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[32] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[33] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[34] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027 [hep-th/0010206].

[35] C.E. Beasley, *BPS branes from baryons*, *JHEP* **11** (2002) 015 [hep-th/0207125].

[36] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of $N = 4$ Yang-Mills from giant gravitons*, *JHEP* **12** (2007) 006 [hep-th/0606087].

[37] G. Mandal and N.V. Suryanarayana, *Counting* 1/8-*BPS dual-giants*, *JHEP* **03** (2007) 031 [hep-th/0606088].

[38] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085].

[39] E. Witten, *Nonabelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455.

[40] D.J. Gross, *Two-dimensional QCD as a string theory*, *Nucl. Phys.* **B 400** (1993) 161 [hep-th/9212149].

[41] D.J. Gross and W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys.* **B 400** (1993) 181 [hep-th/9301068].

[42] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on* 2*D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [hep-th/9411210].